

Underdamped stochastic harmonic oscillator

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We investigate stationary states of the linear damped stochastic oscillator driven by Lévy noises. In the long time limit kinetic and potential energies of the oscillator do not fulfill the equipartition theorem and their distributions follow the power-law asymptotics. At the same time, partition of the mechanical energy is controlled by the damping coefficient. We show that in the limit of vanishing damping a stochastic analogue of the equipartition theorem can be proposed, namely the statistical properties of potential and kinetic energies attain distributions characterized by the same width. Finally, we demonstrate that the ratio of instantaneous kinetic and potential energies which signifies departure from the mechanical energy equipartition, follows universal power-law asymptotics.

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I. INTRODUCTION

A damped harmonic oscillator under influence of noise is one of the fundamental conceptual models in nonequilibrium statistical physics [1–4], broadly used to describe relaxation phenomena in the linear regime. Displacement $x(t)$ from the minimum of the potential $V(x)$ is described by the Langevin equation

$$m \frac{d^2 x(t)}{dt^2} = -\gamma \frac{dx(t)}{dt} - \lambda x(t) + \sqrt{2\gamma \frac{k_B T}{m}} \xi(t), \quad (1)$$

in which the interaction with the environment is separated into deterministic dissipative force $\gamma \frac{dx(t)}{dt}$, describing damping, and a noise term $\xi(t)$ describing all the complexity of the interaction between the test particle (or mode) with the rest of the system. In the simplest situations it is assumed that the noise is white and Gaussian, and averaged over an ensemble of realizations satisfies $\langle \xi(t) \rangle = 0$ and $\langle \xi(t) \xi(s) \rangle = \delta(t - s)$. Here m stands for the particle's (effective) mass, T is the system's temperature, and k_B is the Boltzmann constant. The joint probability density $P = P(x, v, t)$ evolves according to the Smoluchowski-Fokker-Planck equation [5]

$$\frac{\partial P}{\partial t} = \left[-v \frac{\partial}{\partial x} + \frac{\partial}{\partial v} \left(\gamma v + \frac{V'(x)}{m} \right) + \gamma \frac{k_B T}{m} \frac{\partial^2}{\partial v^2} \right] P. \quad (2)$$

The stationary solution of Eq. (2) has the canonical Boltzmann-Gibbs form [5, 6]

$$P(x, v) = C \exp \left[-\frac{1}{k_B T} \left(\frac{mv^2}{2} + V(x) \right) \right], \quad (3)$$

and factorizes, making position and velocity to be statistically independent random variables. If $V(x) = \lambda \frac{x^2}{2}$ as in

Eq. (1), the stationary solution is a 2D, elliptically contoured normal density. Moreover, the average energy of the oscillator in its stationary (equilibrium) state is then given by the classical equipartition value $\mathcal{E} = k_B T$ with $\mathcal{E}_k = mv^2/2$ and $\mathcal{E}_p = \lambda x^2/2$.

$$\left\langle \frac{mv^2}{2} \right\rangle = \left\langle \frac{\lambda x^2}{2} \right\rangle = \frac{k_B T}{2}. \quad (4)$$

This equal distribution of energy among different degrees of freedom can be violated in numerous ways. The simplest option is to assume more general potentials than parabolic ones in Eq. (1). Indeed, for the general single well potentials ($n > 0$)

$$V(x) = \lambda \frac{|x|^n}{n} \quad (5)$$

the stationary density is still of the Boltzmann-Gibbs type with the normalization constant C given by

$$C = 2\sqrt{2\pi} \Gamma \left(1 + \frac{1}{n} \right) \sqrt{\frac{k_B T}{m}} \left(\frac{knT}{\lambda} \right)^{\frac{1}{n}}. \quad (6)$$

In such a case the average kinetic energy is fixed to

$$\langle \mathcal{E}_k \rangle = \left\langle \frac{mv^2}{2} \right\rangle = \frac{k_B T}{2} \quad (7)$$

regardless of the potential type. At the same time the average potential energy depends on the potential type and it is equal to

$$\langle \mathcal{E}_p \rangle = \langle V(x) \rangle = \frac{k_B T}{n}. \quad (8)$$

Another possibility to violate the equipartition of energy is to relax the assumption that noise $\xi(t)$ in Eq. (1) is of the Gaussian white type. The straightforward option is to assume that the noise is still white but of more general (symmetric)

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α -stable Lévy type [7–11]. The white α -stable noise $\zeta_\alpha(t)$, which is a formal time derivative of the α -stable motion $L_\alpha(t)$ [12], results in stochastic increments which are distributed according to the symmetric α -stable density whose characteristic function is given by [7, 8]

$$\phi(k) = \exp[-\sigma^\alpha |k|^\alpha]. \quad (9)$$

The parameter α ($0 < \alpha \leq 2$) is the so called stability index describing asymptotics of α -stable densities which for $\alpha < 2$ is of the power-law type $p(x) \propto |x|^{-(\alpha+1)}$. In the $\alpha = 2$ limit the α -stable noise is equivalent to the Gaussian white noise.

The behavior of a damped harmonic oscillator under the influence of noise with $\alpha < 2$ is very different from those for the Gaussian case $\alpha = 2$. Stationary densities are given by bivariate α -stable densities [7, 13] for which lines of constant probability are not ellipses. Moreover, in the stationary state, velocity v and position x are not statistically independent [7]. Finally, for $\alpha < 2$ there is no equipartition of energy, see Eq. (4).

Motivated by the aforementioned findings, in this paper we investigate measures of departure from equilibrium by analyzing statistical properties of mechanical energy and its distribution in the underdamped linear oscillator. The association of coordinate and velocity is manifested in inhomogeneity of the phase space of the system and contributes to the strongly nonequilibrium properties of non-Gaussian Lévy fluctuations

II. MODEL AND RESULTS

In what follows we examine the distributions of the kinetic and potential energy and of their ratio for the case of a damped harmonic oscillator driven by a white α -stable noise $\zeta_\alpha(t)$:

$$m\ddot{x}(t) = -\gamma\dot{x}(t) - \lambda x(t) + \zeta_\alpha(t), \quad (10)$$

where strength of fluctuations is controlled by the scale parameter σ , see Eq. (9) and its similarity properties are governed by the parameter α . Contrary to the $\alpha = 2$ case, due to the divergence of the second moment $\langle v^2 \rangle$ for $0 < \alpha < 2$, there is no fluctuation-dissipation relation of the Smoluchowski-Sutherland-Einstein type [14–18]. Consequently, damping coefficient γ and fluctuation intensity σ can be viewed as independent parameters. The parameters m and λ of free, undamped oscillator define the most convenient units in which the system can be described. We choose $t_0 = \omega_0^{-1} = \sqrt{m/\lambda}$ to define the unit of time. In the dimensionless time t/t_0 (for a brief explanation of units, cf. Appendix) the equation (10) takes the form

$$\ddot{x}(t) = -\gamma\dot{x}(t) - x(t) + \sigma\zeta_\alpha(t), \quad (11)$$

with a damping γ replacing the original frequency of dissipation $\gamma = \tilde{\gamma}t_0/m$ (for the clarity, we omit the tilde sign over the original constants). Here the prefactor σ of ζ_α , measuring intensity of the noise is $\sigma = \tilde{\sigma}t_0^{1+1/\alpha}/m$. Moreover, in Eq. (11), the white Lévy noise $\zeta_\alpha(t)$ with the scale parameter set to unity is used. The instantaneous kinetic and potential energies of the system are denoted by $\mathcal{E}_k = v^2/2$ and $\mathcal{E}_p = x^2/2$, respectively.

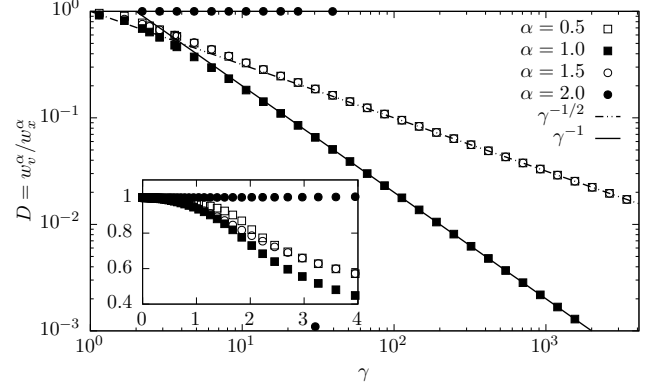


FIG. 1. The quotient of the corresponding prefactors $D = \frac{w_v^\alpha}{w_x^\alpha}$ for various values of the stability index α . Please note the double logarithmic scale in the main plot and the linear scale in the inset.

The formal solution of Eq. (11) is

$$x(t) = F(t) + \int_{-\infty}^t G(t-t')\zeta_\alpha(t')dt', \quad (12)$$

where $G(t)$ is the Green's (response) function of the corresponding process, and $F(t)$ is a decaying function (a solution of the homogeneous equation under given initial conditions). The solution for v is given by

$$v(t) = F_v(t) + \int_{-\infty}^t G_v(t-t')\zeta_\alpha(t')dt', \quad (13)$$

where $G_v(t)$ is the Green's function of the velocity process

$$G_v(t) = \frac{d}{dt}G(t). \quad (14)$$

In a stationary situation, $t \rightarrow \infty$, the F -functions in Eqs. (12) and (13) vanish. The Green's function of Eq. (10) can be easily found e.g. via the Laplace representation, and reads:

$$G(t) = \frac{\exp(-\gamma t/2)}{\sqrt{\omega_0^2 - \gamma^2/4}} \sin \left[\sqrt{\omega_0^2 - \gamma^2/4} t \right] \quad (15)$$

for $\omega_0 = \sqrt{\lambda/m} = 1 > \gamma/2$ (underdamped case),

$$G(t) = t \exp(-\gamma t/2) \quad (16)$$

for $\omega_0 = 1 = \gamma/2$ (critical case) and

$$G(t) = \frac{\exp(-\gamma t/2)}{\sqrt{\gamma^2/4 - \omega_0^2}} \sinh \left[\sqrt{\gamma^2/4 - \omega_0^2} t \right] \quad (17)$$

for $\omega_0 = \sqrt{\lambda/m} = 1 < \gamma/2$ (overdamped case). Note that the functions $G(t)$ vanish both for $t = 0$ and for $t \rightarrow \infty$ so that

$$\begin{aligned} \int_0^\infty G(t) \left[\frac{d}{dt} G(t) \right] dt &= \frac{1}{2} \int_0^\infty \left[\frac{d}{dt} G^2(t) \right] dt \\ &= \frac{1}{2} G^2(t) \Big|_0^\infty = 0, \end{aligned} \quad (18)$$

i.e. $G(t)$ and $G_v(t)$ are orthogonal on $[0, \infty)$.

The characteristic function of the stationary distribution of x and $v = \dot{x}$ is given by Eq. (17) of Ref. [19]:

$$f(k, q) = \exp \left[-\sigma^\alpha \int_0^\infty |kG(t) + qG_v(t)|^\alpha dt \right] \quad (19)$$

where $G(t)$ is the Green's function for the homogeneous part of the equation of motion, see Eqs. (6) – (8) of Ref. [19], $G_v(t) = \frac{d}{dt}G(t)$.

The marginal distributions of x and v have the characteristic functions $f_x(k) = f(k, 0)$ and $f_v(q) = f(0, q)$ and are the Lévy stable ones with index α and scale parameters (widths)

$$w_x^\alpha = \sigma^\alpha \int_0^\infty |G(t)|^\alpha dt \quad (20)$$

and

$$w_v^\alpha = \sigma^\alpha \int_0^\infty |G_v(t)|^\alpha dt. \quad (21)$$

The corresponding integrals can be easily evaluated numerically for small and moderate values of γ for any $\alpha > 0$. Their asymptotic behavior for $\gamma \rightarrow 0$ and for $\gamma \rightarrow \infty$ will be discussed in the next subsection.

The large $|x|$ and $|v|$ asymptotics of the corresponding PDFs are

$$p_x(x) \propto \frac{w_x^\alpha}{|x|^{1+\alpha}} \quad (22)$$

and

$$p_v(v) \propto \frac{w_v^\alpha}{|v|^{1+\alpha}}. \quad (23)$$

A. Ratio of distribution widths

The numerically calculated quotient of the corresponding prefactors

$$D = \frac{w_v^\alpha}{w_x^\alpha} \quad (24)$$

is depicted in the Fig. 1 as a function of the damping coefficient γ . Various curves correspond to different values of the stability index α . The inset shows small γ dependence.

Small ($\gamma \rightarrow 0$) and large ($\gamma \rightarrow \infty$) asymptotics of the quotient D can be calculated analytically. For $\gamma \rightarrow 0$ the system is strongly underdamped, for which

$$G(t) = \frac{\exp(-\gamma t/2)}{\sqrt{1 - \gamma^2/4}} \sin \left[\sqrt{1 - \gamma^2/4} t \right] \quad (25)$$

$$\simeq \exp(-\gamma t/2) \sin t,$$

and

$$G_v(t) \simeq \exp(-\gamma t/2) \left[-\frac{\gamma}{2} \sin t + \cos t \right] \quad (26)$$

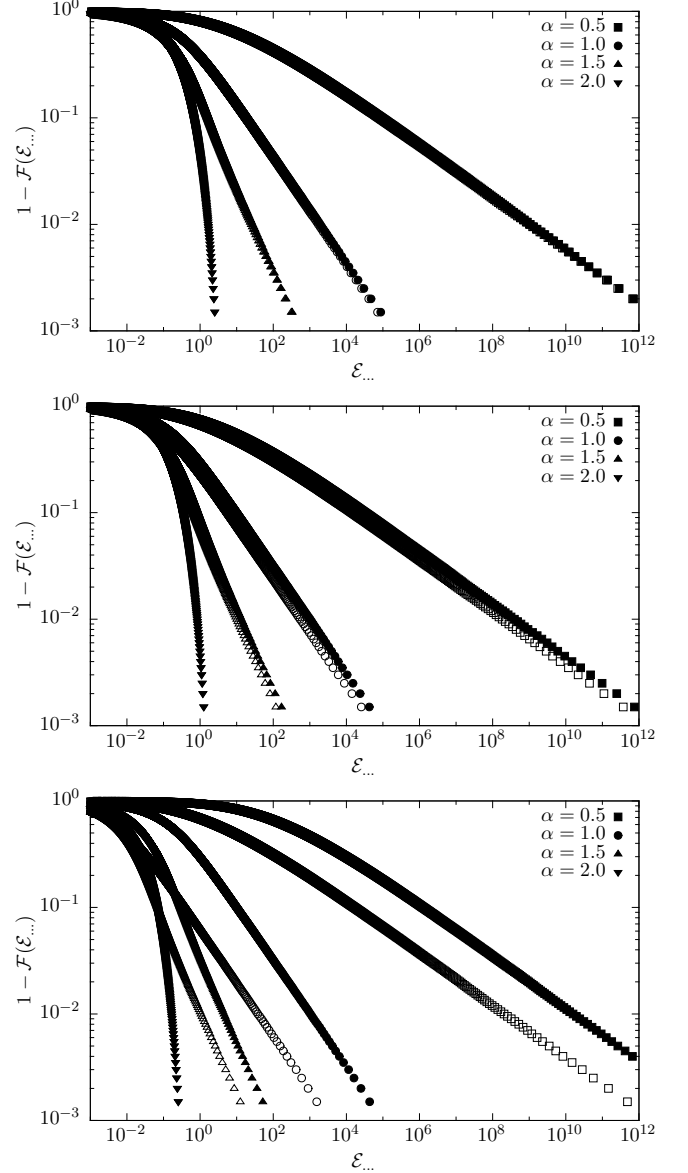


FIG. 2. Distributions of the potential \mathcal{E}_p (full symbols) and kinetic \mathcal{E}_k (empty symbols) energies for $\gamma = 1, 2, 10$ (from top to bottom). Various curves correspond to various values of the stability index α .

in the lowest order in γ everywhere except for the exponential. The last expression can be rewritten as

$$G_v(t) \simeq \exp(-\gamma t/2) \sqrt{1 + \frac{\gamma^2}{4}} \cos(1 + \phi) \quad (27)$$

with $\phi = \arccos(1/\sqrt{1 + \gamma^2/4})$.

For $\gamma \rightarrow 0$ we have

$$w_x^\alpha \simeq \sigma^\alpha \int_0^\infty |\exp(-\gamma t/2) \sin t|^\alpha dt \quad (28)$$

$$= \sigma^\alpha \int_0^\infty \exp(-\gamma \alpha t/2) |\sin t|^\alpha dt,$$

and a similar expression (with a cosine) for w_v^α . For $\gamma \rightarrow 0$ the exponential hardly changes on the period of oscillations

of the trigonometric function, so we can average over these oscillations. Essentially what we do is to split the domain of integration into the π -intervals, which are the domains of periodicity of the absolute value of the trigonometric function and rewrite the total integral as the sum

$$\begin{aligned} w_x^\alpha &= \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \exp(-\gamma\alpha t/2) |\sin t|^\alpha dt \\ &= \sum_{n=0}^{\infty} e^{n\pi\gamma\alpha/2} \int_0^\pi \exp(-\gamma\alpha t/2) |\sin t|^\alpha dt \\ &= \frac{\exp(-\gamma\alpha t^*/2)}{1 - \exp(\pi\gamma\alpha/2)} \int_0^\pi |\sin t|^\alpha dt. \end{aligned} \quad (29)$$

with $0 < t^* < \pi$. The expression for w_v in the lowest order in γ is

$$w_v^\alpha = \frac{\exp(-\gamma\alpha t^{**}/2)}{1 - \exp(\pi\gamma\alpha/2)} \int_0^\pi |\cos(t + \phi)|^\alpha dt, \quad (30)$$

and only differs with respect to the position of the intermediate point $0 < t^{**} < \pi$. The integrals over the trigonometric functions are the same, and are given by

$$\int_0^\pi |\sin t|^\alpha dt = \int_0^\pi |\cos(t + \phi)|^\alpha dt = \sqrt{\pi} \frac{\Gamma[(\alpha + 1)/2]}{\Gamma(\alpha/2 + 1)}. \quad (31)$$

Therefore

$$D = \frac{w_v^\alpha}{w_x^\alpha} = \exp[-\gamma\alpha(t^{**} - t^*)/2] \rightarrow 1 \quad (32)$$

for $\gamma \rightarrow 0$. For small friction we have a “kind of” equipartition, i.e. both densities $p(x)$ and $p(v)$ are characterized by the same width. In the lowest order in γ this can be obtained by expanding the denominator:

$$w_v^\alpha = w_x^\alpha = \frac{2}{\sqrt{\pi}\gamma\alpha} \frac{\Gamma[(\alpha + 1)/2]}{\Gamma(\alpha/2 + 1)}. \quad (33)$$

For $\gamma \rightarrow \infty$ we start from the explicit expression

$$\begin{aligned} G(t) &= \frac{\exp(-\gamma t/2)}{\sqrt{\gamma^2/4 - 1}} \sinh \left[\sqrt{\gamma^2/4 - 1} t \right] \\ &= \frac{1}{2\sqrt{\gamma^2/4 - 1}} \left[e^{(\sqrt{\gamma^2/4 - 1} - \gamma/2)t} - e^{-(\sqrt{\gamma^2/4 - 1} + \gamma/2)t} \right] \\ &\simeq \frac{1}{\gamma} \left(e^{-\frac{t}{\gamma}} - e^{-\gamma t} \right) \end{aligned} \quad (34)$$

where in the last line only the leading contributions in γ in the exponential are retained. For $\gamma \rightarrow \infty$ the first exponential is decaying very slowly, and gives the major contribution to the time integral

$$G(t) \simeq \frac{1}{\gamma} e^{-\frac{t}{\gamma}}. \quad (35)$$

We now can estimate the integral for w_x^α , and get

$$w_x^\alpha \simeq \frac{\gamma^{1-\alpha}}{\alpha}. \quad (36)$$

Similarly for $G_v(t) = \frac{d}{dt}G(t)$ we get

$$G_v(t) \simeq \frac{1}{\gamma^2} e^{-\frac{t}{\gamma}} + e^{-\gamma t}. \quad (37)$$

In order to calculate w_v^α special care is required. The second term in Eq. (37) cannot be neglected because for $\gamma \gg 1$ it is larger than the first one

$$w_v^\alpha = \int_0^\infty |G_v(t)|^\alpha dt = \int_0^\infty \left[\frac{1}{\gamma^2} e^{-\frac{t}{\gamma}} + e^{-\gamma t} \right]^\alpha dt. \quad (38)$$

The integrand (38) shows a crossover between two types of asymptotics: for $\gamma \gg 1$ the short time behavior is dominated by the fast decay $e^{-\alpha\gamma t}$, while at long times it is dominated by the slow decay $\gamma^{-2\alpha} e^{-\alpha t/\gamma}$. The crossover between two regimes takes place at time $t_c = 2\gamma^{-1} \ln \gamma$. Consequently, the total integral can be estimated as

$$w_v^\alpha \simeq \int_0^{t_c} e^{-\alpha\gamma t} dt + \gamma^{-2\alpha} \int_{t_c}^\infty e^{-\frac{\alpha t}{\gamma}} dt \quad (39)$$

resulting in

$$w_v^\alpha \simeq \frac{1}{\alpha\gamma} \left[1 - \frac{1}{\gamma^{2\alpha}} \right] + \frac{\gamma^{1-2\alpha}}{\alpha} \gamma^{2\alpha/\gamma^2}. \quad (40)$$

For $\gamma \gg 1$ the above expression can be further simplified to

$$w_v^\alpha \simeq \frac{1}{\alpha\gamma} + \frac{\gamma^{1-2\alpha}}{\alpha} \quad (41)$$

leading to the dominating terms

$$w_v^\alpha \simeq \begin{cases} \frac{\gamma^{1-2\alpha}}{\alpha} & \text{for } 0 < \alpha < 1 \\ \frac{1}{\alpha\gamma} & \text{for } 1 \leq \alpha \leq 2 \end{cases}. \quad (42)$$

Finally, the ratio of distributions widths scales as

$$D = \frac{w_v^\alpha}{w_x^\alpha} = \begin{cases} \gamma^{-\alpha} & \text{for } 0 < \alpha < 1 \\ \gamma^{\alpha-2} & \text{for } 1 \leq \alpha \leq 2 \end{cases}. \quad (43)$$

For $\alpha = 2$, from Eq. (34) and the definition

$$D = 1, \quad (44)$$

as predicted by Eq. (43). Numerical simulations perfectly confirm the scaling predicted by Eq. (43), see Fig. 1. Please note, that results for $\alpha = 0.5$ (empty squares) and $\alpha = 1.5$ (empty circles) coincide.

B. Energy distributions

Let us consider the distributions of the kinetic and the potential energies. Through the change of variables $x = \pm\sqrt{2\mathcal{E}_p}$, $v = \pm\sqrt{2\mathcal{E}_k}$ we get

$$p(\mathcal{E}_p) = p_x(\sqrt{2\mathcal{E}_p}) \mathcal{E}_p^{-1/2} \propto \frac{w_x^\alpha}{\mathcal{E}_p^{1+\frac{\alpha}{2}}}, \quad (45)$$

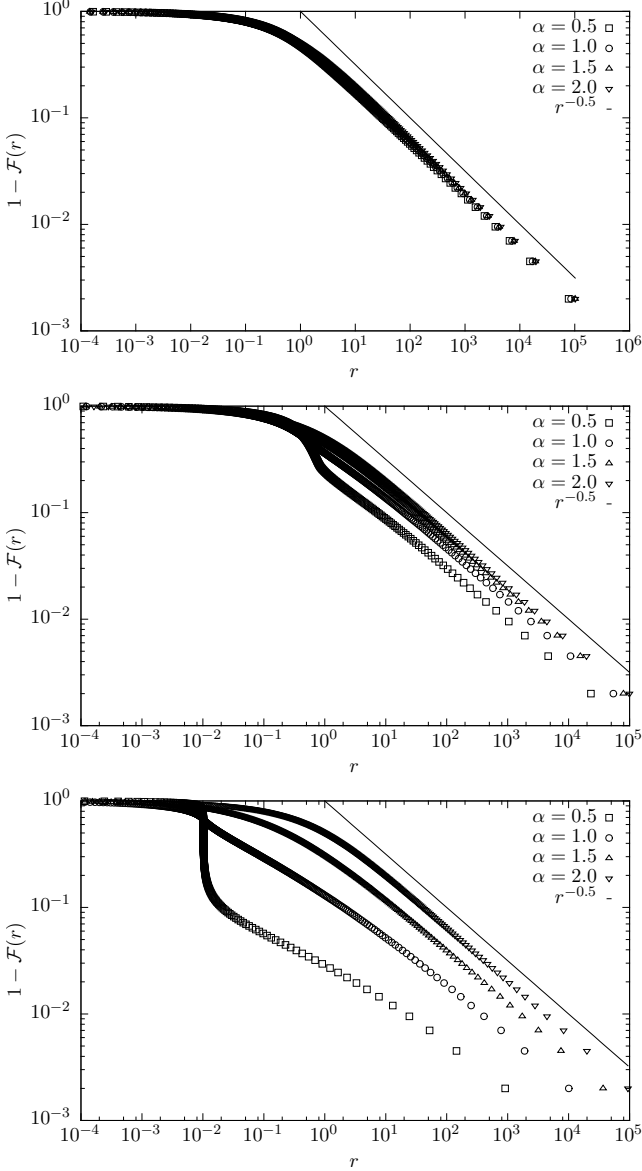


FIG. 3. Distribution of the energy ratio $r = \mathcal{E}_k/\mathcal{E}_p$ for $\gamma = 1, 2, 10$ (from top to bottom).

$$p(\mathcal{E}_k) = p_x(\sqrt{2\mathcal{E}_k})\mathcal{E}_k^{-1/2} \propto \frac{w_v^\alpha}{\mathcal{E}_k^{1+\frac{\alpha}{2}}}. \quad (46)$$

The total energy \mathcal{E} in our units is proportional to the square of the amplitude of the phase space vector $A = \sqrt{x^2 + v^2}$: $E = A^2/2$. Since this amplitude has a distribution

$$p(A) \propto \frac{1}{A^{1+\alpha}}, \quad (47)$$

the total energy has the same asymptotics as the kinetic or the potential one, and it is dominated by the potential energy in the case of large γ .

Figure 2 presents sample potential \mathcal{E}_p (full symbols) and kinetic \mathcal{E}_k energy (empty symbols) distributions for various values of the damping coefficient γ ($\gamma = 1, 2, 10$) and stability index α ($\alpha = 0.5, 1.0, 1.5, 2.0$).

It is interesting to discuss the behavior of the instantaneous quotient of the kinetic and potential energy

$$r = \frac{\mathcal{E}_k(t)}{\mathcal{E}_p(t)} = \frac{v^2(t)}{x^2(t)}. \quad (48)$$

This property of the system is closely connected with another phase space property, namely with the phase angle

$$\phi = \arctan \frac{v}{x} \quad (49)$$

The distribution $p(r)$ has a universal asymptotics, and this universality is closely connected with non-independence of the velocity and coordinate processes [19].

Imagine the distribution of the phase angle is known and is given by a PDF $p(\phi)$. Then $r = \tan^2 \phi$ and

$$\begin{aligned} p(r) &= p_\phi(\pm\phi(r)) \left| \frac{d\phi}{dr} \right| \\ &= [p_\phi(\arctan \sqrt{r}) + p_\phi(-\arctan \sqrt{r})] \frac{1}{2(1+r)\sqrt{r}}, \end{aligned} \quad (50)$$

note that there are two solutions for ϕ for a given r . For $r \rightarrow \infty$ the argument of p_ϕ , $\arctan \sqrt{r}$ tends to $\pi/2$, and therefore the behavior of $p(r)$ depends on whether $p_\phi(\phi)$ does or does not have a singularity at $\phi = \pm\pi/2$.

In the case of a driven harmonic oscillator, the association between the velocity and the coordinate processes makes the distribution $p_\phi(\phi)$ non-singular at $\pm\pi/2$, which can be seen from the expressions of the corresponding spectral measures as given in [20] which do not show singularities at $\theta = \pm\pi/2$, and can be grasped from the graphical representation of the corresponding level curves. Therefore for $r \rightarrow \infty$

$$p(r) \propto \frac{C}{(1+r)\sqrt{r}} \simeq \frac{1}{r^{3/2}} \quad (51)$$

with $C = [p_\phi(-\pi/2) + p_\phi(\pi/2)]/2$, and is independent of α . Interestingly enough, the inverse of r , the ratio of the potential and the kinetic energy, has exactly the same asymptotic distribution, as it is evident by the explicit change of variables.

If x and v were independent the behavior would be strongly different, due to the fact that the corresponding spectral measure is concentrated (i.e. has singularities) at $\theta = 0, \pm\pi/2$, and π , i.e. at the intersections of the unit sphere with the axes [7]. The distribution of r in this case would be very different, and can be derived from the distribution of a quotient of two independent symmetric Lévy-stable variables which possesses quite a complicated form [21].

The qualitative discussion of the asymptotic behavior of r is however very simple. The large values of

$$z = \frac{x}{y} \quad (52)$$

will typically appear either due to the very large values of the numerator or to very small values of denominator (the possibility that both occurs simultaneously is very small, and plays

the role only in the case of the Cauchy distribution, *vide infra*). In the symmetric Lévy case the probability density of large values of numerator decays as

$$p(x) \simeq \frac{1}{|x|^{1+\alpha}}, \quad (53)$$

while the distribution of $q = 1/y$ is given by the variable transformation

$$p(q) = p\left(\frac{1}{q}\right) \frac{1}{q^2}. \quad (54)$$

Since $p(x)$ is non-singular and does not vanish at zero, the tail of the PDF $p(q)$ is universal and of the same type as for the Cauchy distribution:

$$p(q) \propto \frac{1}{q^2}. \quad (55)$$

For $\alpha > 1$ the tail of $p(z)$ is dominated by the tail of $p(q)$ and therefore $p(z) \propto z^{-2}$. The variable transformation to $r = z^2/2$ transforms this tail into

$$p(r) = \frac{1}{r^{3/2}}, \quad (56)$$

exactly as in the case of the correlated variables above. An explicitly solvable example is given by the Gaussian case $\alpha = 2$ for which the distribution of z is known explicitly: it is a Cauchy distribution

$$p(z) = \frac{1}{\pi} \frac{1}{1 + z^2}. \quad (57)$$

The variable transformation to r gives

$$p(r) = \frac{1}{\pi} \frac{1}{(r+1)\sqrt{r}}. \quad (58)$$

For $\alpha < 1$ the tail of the ratio z is dominated by the tail of the denominator, so that

$$p(z) \simeq \frac{1}{|z|^{1+\alpha}}, \quad (59)$$

and the variable transformation gives

$$p(r) \simeq \frac{1}{r^{1+\frac{\alpha}{2}}}. \quad (60)$$

The transition between the two regimes happens at $\alpha = 1$, i.e. for the Cauchy distribution, for which the distribution of the ratio of the two variables is again explicitly known [22]

$$p(z) = \frac{1}{\pi^2} \frac{1}{z^2 - 1} \ln z^2, \quad (61)$$

so that

$$p(r) = \frac{1}{2\pi^2} \frac{1}{(r-1)\sqrt{r}} \ln r \quad (62)$$

and involves a logarithmic correction: its asymptotic behavior is

$$p(r) \simeq \frac{\ln r}{r^{\frac{3}{2}}}. \quad (63)$$

Consequently, if the velocity v and position x would be independent, the ratio of instantaneous kinetic \mathcal{E}_k and potential \mathcal{E}_p energy, has the following non-universal asymptotics

$$p(r) \simeq \begin{cases} \frac{1}{r^{1+\alpha/2}} & \text{for } 0 < \alpha < 1 \\ \frac{\ln r}{r^{\frac{3}{2}}} & \text{for } \alpha = 1 \\ \frac{1}{r^{3/2}} & \text{for } 1 < \alpha \leq 2 \end{cases}. \quad (64)$$

Contrary to this α -dependent behavior, the asymptotic behavior of $p(r)$ for the harmonic Lévy oscillator, where v and x are not independent, is universal:

$$p(r) \simeq r^{-3/2}. \quad (65)$$

Figure 3 presents the ratio r of instantaneous kinetic \mathcal{E}_k and potential \mathcal{E}_p energy for $\gamma = 1, 2, 10$ (from top to bottom). Various panels correspond to different values of the stability index α .

III. SUMMARY AND CONCLUSIONS

Stationary states for a particle moving in the parabolic potential driven by the white Lévy noise reflect symmetries of the noise, i.e. they are given by the α -stable densities both for symmetric [23] and asymmetric noises [24]. The same effect is observed for a 2D parabolic potential perturbed by the bi-variate Lévy noise [25].

In the present work we extend earlier studies of the damped harmonic oscillator driven by Lévy noises in phase space characterized by the position x and velocity $v = \dot{x}$ [19]. The stationary state is given by a 2D α -stable density [7]. First of all, in stationary states, position and velocity are no longer independent [19], leading to considerable difference from the usual case of the Gaussian white noise. Properties of stationary states are controlled by damping coefficient γ . In the present work we concentrate on the distributions of the kinetic and potential energy of the stochastically driven oscillator.

Kinetic and potential energies of a harmonic oscillator driven by a symmetric α -stable noise have the same power-law asymptotics of $\mathcal{E}^{-(1+\alpha/2)}$ type determined by the noise type. Contrary to the classical Gaussian case, showing the equipartition between the kinetic and the potential energy, $\langle \mathcal{E}_k \rangle = \langle \mathcal{E}_p \rangle$, no such equipartition in a whatever statistical sense is observed for the Lévy noise, except for the case of vanishing damping. For small friction there is a “kind of” stochastic equipartition, i.e. the potential and the kinetic energy distributions in the limit of $\gamma \rightarrow 0$ have the same widths. With the increasing damping larger fraction of energy is stored in the form of the potential energy. Consequently, with increasing γ the ratio of kinetic and potential energy distributions’ widths decreases, and is given by a power-law in γ with

the exponent depending on the stability index α . In the limit of $\gamma \rightarrow \infty$ the system is fully overdamped. In such a case stochastic oscillator is fully characterized by its position only [23], and the kinetic energy vanishes.

Finally, we have studied the distribution of the ratio of instantaneous kinetic and potential energies in the stationary state. We show that this ratio has a universal $r^{-3/2}$ asymptotics independent on α , which differs strikingly from the situation when the position and the velocity of the oscillator were independent.

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V. APPENDIX

The white Lévy noise $\zeta_\alpha(t)$ is by definition a formal derivative of the strictly α -stable Lévy motion process $\zeta_\alpha(t) \equiv \frac{dL_\alpha(t)}{dt}$ whose increments are independent stationary variables distributed according to the symmetric α -stable density, see Eq. (9). The self-similarity of the Lévy motion (Lévy process) signifies that its realizations fulfill the condition $L_\alpha(t) = t^{1/\alpha} L_\alpha(1)$. In view of the above, scaling properties of the Lévy white noise assume the change of variables according to $\zeta_\alpha(t_0 t) \rightarrow t_0^{1/\alpha-1} \zeta_\alpha(t)$.

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